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Count-on: The Parting of the Ways for Simple Arithmetic

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This paper considers the pivotal role that "count-on" plays in the development of qualitatively different forms of thinking in simple arithmetic. Viewed as a compression of the "count-all" procedure, "count-on" is seen as one example of a procedure that leads to a bifurcation in mathematical thinking between those who operate flexibly with ***procepts*** *(in the sense of Gray & Tall, 1991) and those who use inflexible* ***procedures****. Qualitative and quantitative evidence is presented which exhibits the diverging strategies of children being tested in the early stages of the British National Curriculum in Mathematics (D.E.S., 1989). This shows that children can attain the same level and yet some may be operating in a different way which inhibits subsequent success.*

Diverging approaches to basic arithmetic have been identified by a number of authors (e.g. Carpenter, Hiebert & Moser, 1981; Gray, 1991) and often these provide an indication as to whether the child operates flexibly with number or uses inflexible procedures. This paper considers a point in development when the parting of the ways may occur. Evidence is offered, both qualitative and quantitative, to suggest that a significant divergence occurs from the application of the "count-on" procedure for addition: it may simply be a procedure to solve any given addition problem, or a means through which new number knowledge is developed. The contention is that for the latter to happen the concept of number requires a flexible meaning. Gray & Tall (1991) used the word "procept" to describe a symbol such as 3 + 2 which could evoke both a process (of addition) and a concept (of sum). Number as a procept may be composed and decomposed in many flexible ways: 4 + 4 is 8, so 4 + 5 is "one more", which is 9. In the context of such flexible thinking, "count-on" may be viewed as a process to build up relationships between flexible number procepts. But if the thinking is only procedural, then "count-on" is simply a procedure to be carried out in time to solve a given addition problem with each problem seen in isolation. Subtraction problems are treated very differently. In proceptual terms, if 16 plus something is 18, then the "something" must be 2, so subtraction is another way of viewing addition facts. In procedural terms, the reverse of the "count-on" procedure is "count-back". Counting back 16 starting from 18 is a far more difficult operation to carry out. Therefore the difference between seeing "count-on" as a procedure and as a means of obtaining flexible number knowledge may be the parting of the ways between an increasingly inflexible method in which arithmetic becomes more and more difficult, and a flexible method which can lead to long-term success.

Retrieving the solution to number combinations

There appears to be general agreement that, over time, there is gradual change in the way many children handle basic number combinations. Although in essence this change is manifest by a decline in counting methods and an increase in fact retrieval methods, the issue of how responses are made to basic number combinations is a contentious one which generates models that focus either on individual fact retrieval (e.g. Thorndike, 1922; Siegler & Shrager, 1984;), or on both individual fact retrieval and rule and procedure generated responses (e.g. Baroody & Ginsburg, 1986).

The place of counting in the development of number awareness is unquestionable, and it is common, certainly in England, for children to be exposed to a variety of number combinations in the belief that by solving them through counting they may 'understand' addition and subtraction. However, an implicit objective behind such exposure and practice is that children will learn the combinations they have practised. Ashcraft (1985) indicated that incidental learning would predict the eventual memory of most combinations but Steffe, Richard & von Glaserfeld (1981) point out that a problem with practice and exposure methods is that factual knowledge, viewed in the context of operations involving counting, would seem to involve reflective abstraction. Carpenter (1986) implicitly supports this notion by indicating that learned procedures may not ensure that related conceptual knowledge has been acquired.

The Compression of Counting Procedures.

This relationship between the ability to use a learned counting procedure such as "count-all", or a compressed procedure such as "count-on", and the ability to recall or "derive" the solution to a number combination, brings us face to face with the procedural/conceptual interface, an issue addressed by Hiebert & Lefevre (1986). For Gray & Tall (1991, 1992) such an interface is a cognitive manifestation of the characteristics inherent in mathematical symbolism. They analyse the various counting procedures in proceptual terms.

"Count-all" is seen as three counting procedures; count one set, count another, put the sets together and count that. Its inverse, "take away", carries out the reverse process of counting the whole set, counting a part to be removed and counting what is left. As "count-all" occurs in time as three distinct procedures, it is hypothesised that this will not easily lead to the development of flexible known facts.

"Count-on", although superficially a compression of "count-all", is a sophisticated double-counting in which 4 + 2 involves counting "five, six" at the same time as keeping track that two numbers have been counted. The inverse of "count-on" is "count-back", starting at the larger number and reciting the number sequence backwards to count off the number to be subtracted. Although "count-back" may be seen as a compression of the take away procedure, it also involves double counting but the counting goes in opposite directions: incrementing and decrementing in ones at each count. To do this children generally need some form of counting aid to maintain a check of the amount counted back. An alternative approach is "count-up" but this involves recognition of a proceptual link between addition and subtraction. The relationship between a number triple $5 - 3 = \square$ can be identified by incrementing using a procedure in the same way that $3 + \square = 5$ can be solved by incrementing with a procedure.

Here, however, it is conjectured that a proceptual link has been recognised between the subtraction of three from five and the addition of something to the three to make five.

Count-on: The Parting of the Ways

It is conjectured that the use of "count-on", and its complementary procedures of "count-up" and "count-back", leads to a divergence between those who apply flexible thinking through the use of procepts and those who think procedurally. Such a divergence has been termed the proceptual divide (Gray & Tall, 1991). Figure 1 places "count-on" in a pivotal role prior to this parting of the ways.

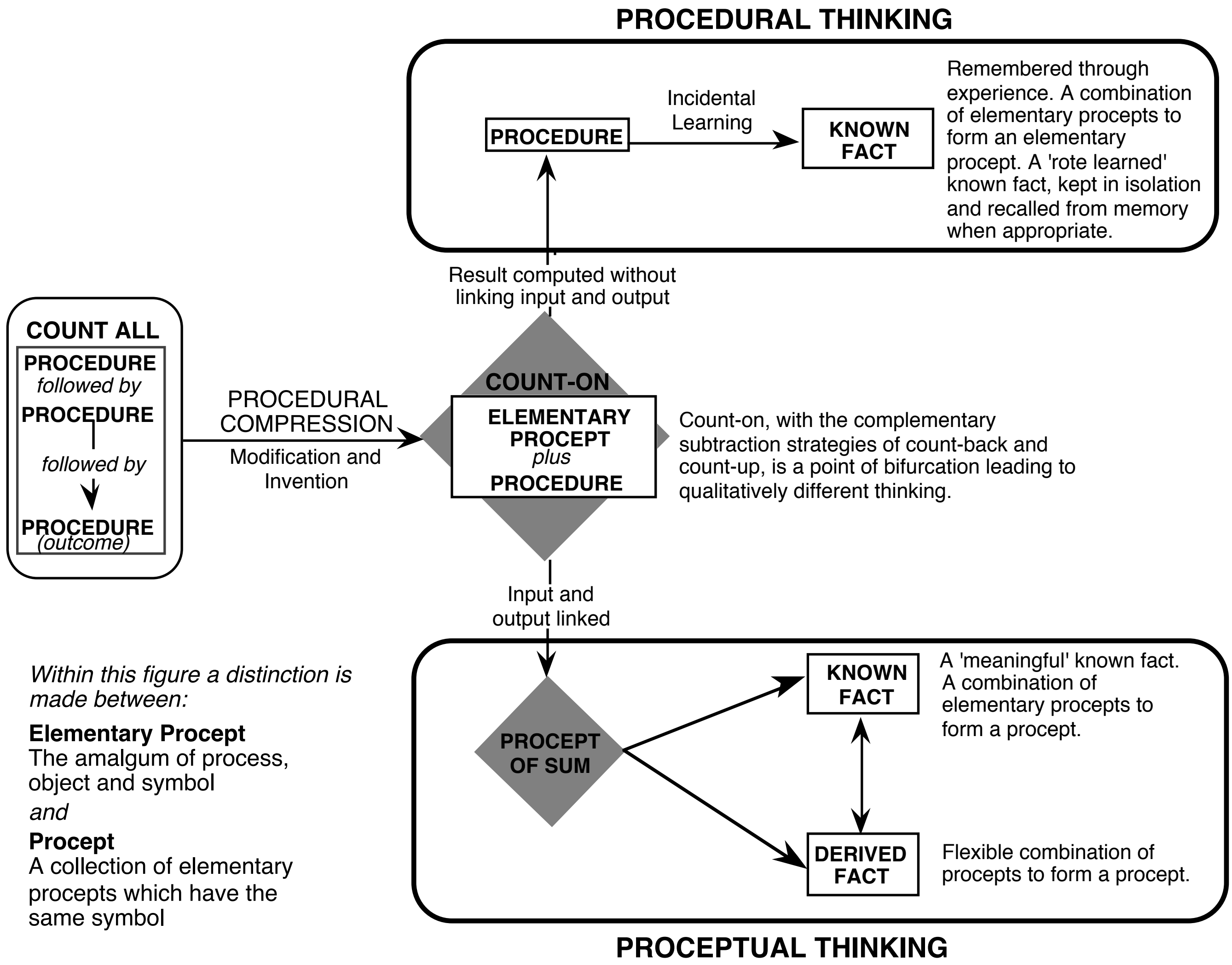

Figure 1: Count-on: The pivot for a proceptual divide in simple arithmetic.

The cognitive compression of the "count-all" strategy into "count-on" implies a compression in the number of procedures used. It does not imply compression in the time taken to implement the procedure. This may happen if the physical or mental supports used are suitable and then "count-on" may have the potential for reflective abstraction. If the inputs and the outputs which are the results of the incremental or decremental procedures are linked there is a possibility of reflection; 3 + 5 = 8 may now be seen as the procept of sum. Of course this may be remembered as an isolated piece of factual knowledge and herein lies the problem of identifying whether of not a known fact is rote learned or whether it is a proceptually known fact–the distinction made between an elementary procept and a procept (Gray & Tall, 1992).reflects this observed cognitive reality. In isolated instances the answer to this question is not easily resolved. Indeed resolution may only come with evidence of one known fact being used to form another, but, this could be almost instantaneous! The distinction between proceptual thinking and procedural thinking may be identified through the integrated use of known facts, derived facts and procedures on the one hand and the extensive use of procedures and isolated known facts on the other. It is through the absence of the ambiguity that we may identify the proceptual divide. From a procedural point of view the essence of strategies such as “count-on”, “count-back” or “count-up” is that they may be refined to such a degree that though they may become very efficient at one level they may not only mitigate against reflection but also against success at the next higher level.

Evidence for the bifurcation caused by count-on

Method

Evidence for the parting of the ways arises from the analysis of the responses made by a class of mixed ability children (N=29), aged between 6 years 8 months and 7 years 7 months, in the numerical components of a series of Standard Assessment Tasks (SAT), (SEAC 1992) allied to the National Curriculum of England and Wales. The tests were administered during the summer term of 1992. The numerical components were part of a broader spectrum of Mathematics Assessment Tasks (MAT) which included the option of Data Handling and Probability.

The numerical components included addition and subtraction number combinations. The maximum time allowed for each item was five seconds but, for a child to achieve a particular level of attainment, only one error in addition and one error in subtraction was allowed. As a result of their responses children were identified as having the following levels of achievement:

- Level 1 (L1): Could add and subtract objects where the numbers involved were no greater than ten.
- Level 2 (L2): Achieved the above and illustrated that they were able to recall the number combinations to ten without calculation.
- Level 3 (L3): Achieved the above and illustrated that they were able to recall the number combinations to twenty without calculation.

The children were recorded on camera during the formal elements of the tests, where problems were presented orally, and then interviewed separately in a second interview, where problems were presented orally and through written symbols, within three weeks of the formal testing.

Results

The analysis of both test video and interview video indicates that for many of the children, even though the purpose of the time limit was to prevent calculation, counting was the dominant means through which solutions were obtained.

Children who were unsuccessful at L2, not only failed to recall the solutions to most of the number combinations but then attempted to use a procedure which was either inefficient or too lengthy to satisfy the timed criteria. For example, when attempting the formal component Simon tried to obtain solutions through "count-all" using his fingers as a procedural anchor. His procedure was so inefficient and lengthy that he not only ran out of time to obtain a solution, but, his concentration on its application inevitably meant that he also failed to hear the first part of the subsequent combination. Joseph, on the other hand, tried to carry out all of the counting in his head with no external physical support. There is no evidence to indicate whether he was attempting count-all or count-on. Whichever, he found his strategy very hard and would sit for extended periods with no obvious sign of action but he, "*...liked trying to do things my head. I like them to be harder because when I grow up I will be able to do harder things*." Joseph failed to obtain the solution to any of the L2 addition or subtraction combinations.

When they counted, most of the children successful at L2 used their fingers to support a "count-on" procedure. Frequently a subitised display of fingers equivalent to the second set was used to maintain a check on the amount counted. When subtracting, many children used an approach which involved the immediate display of the large amount through extending a number of fingers equivalent to it, immediately curling a number of fingers equivalent to the value of the small set and the subitising of the value of the remaining extended fingers. This **enactive subitising** involved no actual counting, the child seemed to need visual support of the numbers in a concrete form; to see that, three add five equals eight.

At the formal level, achieving L2 took no account of the means by which the level was attained; there was no differentiation between those who extensively used counting and those who solved every combination by recalling the solution to the addition and subtraction combinations. Only when the children began to attempt the L3 of the MAT that the real differences began to emerge.

Many children who failed to achieve the L3 level of attainment not only knew very few of the combinations to twenty but also attempted to use a procedure which they could not generalise within the time limit. The general pattern that emerged from those who achieved L2, but failed to achieve L3, was that they recalled solutions to combinations such as 17 + 0, 7 + 7 and 15 – 0 but for all of the others i.e. 9 + 6, 4 + 11, 15 + 2, 17 – 6, 11 – 9, and even 18 – 10 they attempted to use a counting procedure. Children successful at L3 not only knew many combinations but their solution of others demonstrated a considerable degree of flexibility through the use of alternative approaches: 9 + 6, "*You get nine, add one to make ten, and then add five*."; 19 – 13, "*You have thirteen and count on to the nineteen–you add some of the nineteen onto the thirteen*."; 11 – 9, "*I took one away from the eleven. That leaves ten. You take one away from that and that leaves nine*." This flexibility also included the efficient use of a procedure. Jonathan, for example, recalled all of the solutions to the combinations apart from two. To obtain the solution to 17 – 6 he counted back six from seventeen. When attempting 20 – 5 he knew that "*fifteen add five is twenty so twenty take away five is fifteen*". However, though it was not unusual for children who achieved L3 to demonstrate a limited use of counting some recognised its limitations. They did not use it if they felt it was an unreasonable approach. Anthony didn't do 19 – 13 because *"...it was a bit too hard and I knew I couldn't count it in quickly"*.

Figure 2 illustrates the overall percentage of number combinations attempted through the use of a counting procedure whilst figure 3 shows the percentage of occasions when these attempts counted towards the achievement of a particular level of attainment.

Of particular interest in figures 2 and 3 is the overall extent with which counting was used and the extent with which it led to success. The difference between those who did not achieve L2 and those who achieved L3 is particularly striking. *Not only did the latter use counting considerably less frequently but when they used it they did so with more success than children within the other groups.*

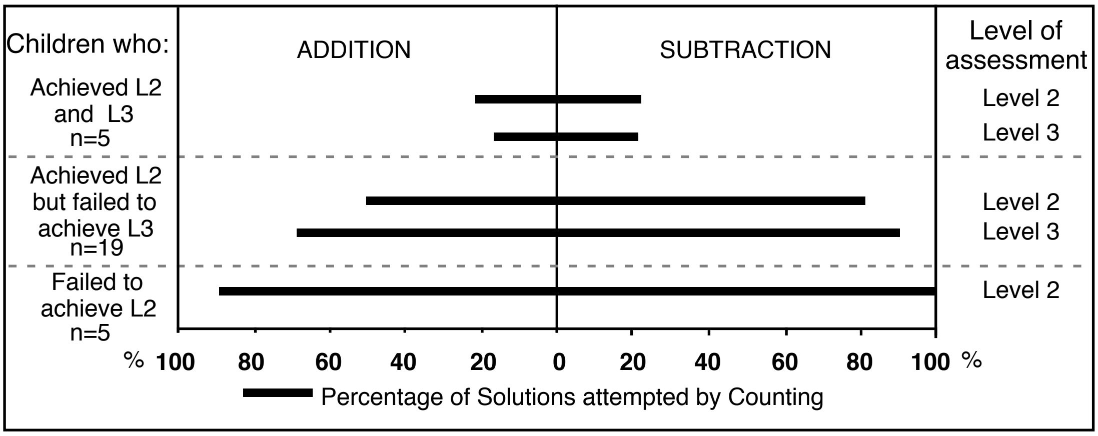

Figure 2: Percentage of solutions to basic number combinations attempted through counting

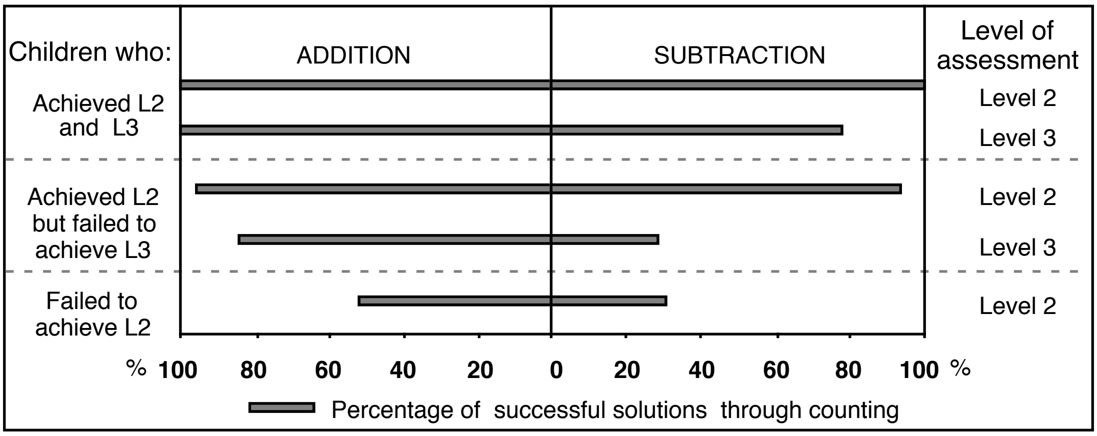

Figure 3: The percentage of successful solutions obtained through counting.

For the children who failed to achieve L2 and for those who failed to achieve L3 the focus appeared to be on the *doing* aspect of the arithmetic. An inability to recall the solution to a number combination appeared to turn their attention to the identification of an action which would enable them to arrive at a solution; their concentration was then on the implementation of this action. The more successful children demonstrated that they had alternatives available to them; through this flexibility they either established the solution to a combination through a derived fact or used a very efficient counting procedure.

Conclusion

In one sense the MAT component relating to the knowing and using number facts achieved its purpose of differentiating between children over three levels. However a longer term prognosis, which only takes note of a child's current level of attainment as a starting point without noting how the attainment was achieved, may lead to a very different outcome.

Clearly, asking children to attempt what is considered to be the same range of problems presents each child with a different level of difficulty. This is not only dependant upon what they know and the way in which they use what they know but it is also a function of their procedural approach and particularly:

- The frequency with which the procedure was used by an individual child at the lower level,
- The efficiency of the procedure,
- The ease with which the procedure can be generalised,
- The ability to recognise the limitations of the procedure.

In the context of simple arithmetic, it is believed that at some point all children use "count-all" but within a varying amount of time this is compressed into "count on". "Count-on" provides them all with not only a potentially more efficient procedure to handle problems but it also acts as a springboard to a different quality of thinking. In this sense it acts as a "junction box"; it can cause a bifurcation that leads to a parting of the ways between those who are successful and those who are not successful. The evidence from the sample points to what is happening at the time of bifurcation. In the longer term it is hypothesised that many of the L2 children will develop a proceptual view of simple arithmetic. Through their experience of "count on" they will move to an even faster track—that provided by proceptual thinking– which provides them with greater flexibility (see for example, Gray, 1991). Other children may oscillate between the relative speed obtained through fairly efficient "count-on" procedures, and, in isolated instances, attempts at proceptual thinking. Gray & Tall (1992) illustrate examples of below average ability children who use derived facts to obtain some solutions to basic number combinations. In some cases their use is almost procedural i.e. "*when I have to add 4 and 5 I always say two five's and then take away one" (*Thomas, age 11). In other cases the derivation is so cumbersome that eventually it is felt that the child will stop using the approach and resort to procedural methods.

The difficulties that children have in establishing a proceptual view of simple arithmetic should be a signal us to the difficulties they may have in more complex areas of arithmetic. How may we reasonably expect children to understand the multiplication and division if in simple addition and subtraction that are still procedural? Difficulties that children have in establishing a proceptual view of simple arithmetic should be a signal to us of the difficulties they will have in developing a proceptual view of fractions. However there is a big difference between the two. Whilst in simple arithmetic it is *advantageous* for the child to think proceptually, when operating with rational numbers it is *incumbent* upon them to do so. For the child to operate the addition and subtraction of fractions successfully they need to be able to see the same fraction in many different ways.

It is conjectured that the problem of the proceptual divide that occurs in simple arithmetic is a microcosm of the problems that occur as mathematics becomes more complex; at each higher level a proceptual divide occurs. Some children take to the fast route fairly easily to become successful, others take the slower, procedural route to achieve success at one level only to be faced with another parting of the ways through which they take an even slower route which eventually leads to failure.

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